Fragments of Second Order Propositional Logic (Proof theoretical study of the structure of logic and computation)

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abstract

This paper shows that the $\wedge\neg\exists\forall$ fragment of classical logic is equivalent to the same fragment of intuitionistic logic, where $\forall$ and $\exists$ are second order quantifiers for propositional variables.

1 Introduction and Definition

1.1 Introduction

This paper shows that the $\wedge\neg\exists\forall$ fragment of classical logic is equivalent to the same fragment of intuitionistic logic, where $\forall$ and $\exists$ are second order quantifiers for propositional variables. Since our framework is the sequent calculus, it can be written as follows.

$$\mathbf{LK}_{\wedge\neg\exists\forall} \vdash \Rightarrow A \iff \mathbf{LJ}_{\wedge\neg\exists\forall} \vdash \Rightarrow A$$

Tatsuta, Fujita, Hasegawa and Nakano showed the case of the $\wedge\neg\exists$ fragment by using the natural deduction in [1]. Recently they showed also the case of $\wedge\neg\exists\forall$ fragment independently in [2].

In this paper, we show the case of the $\wedge\neg\exists$ fragment in the section 2. We construct a tree which is associated with a proof of $\mathbf{LK}_{\wedge\neg\exists}$ and translate it to a proof of $\mathbf{LJ}_{\wedge\neg\exists}$. Next, we show the case of the $\wedge\neg\exists\forall$ fragment in the section 3. Finally, we consider the case of $\wedge\vee\neg\exists\forall$ fragment in the section 4. In this case, $\mathbf{LK}_{\wedge\vee\neg\exists\forall}$ and $\mathbf{LJ}_{\wedge\vee\neg\exists\forall}$ are not equivalent.We show partial equivalency of them. If formulas are restricted properly, $\mathbf{LK}_{\wedge\vee\neg\exists\forall}$ and $\mathbf{LJ}_{\wedge\vee\neg\exists\forall}$ are equivalent.

1.2 Notation

In this paper, $p, q, r, \ldots$ denote propositional variables. $\top$ and $\bot$ are propositional constants. $A, B, C, \ldots$ denote formulas, and $\Gamma, \Delta, \Sigma, \ldots$ denote finite multi sets of formulas.

1.3 Formulas

Formulas are defined by

- $\top, \bot, p, q, r, \ldots$ are (prime) formulas.
- If A and B are formulas, $A \wedge B, A \vee B, \neg A, \exists p A$ and $\forall p A$ are formulas.

$A \veebar B$ is an abbreviation of $\neg(\neg A \wedge \neg B)$. $\forall \bar{p} A$ is an abbreviation of $\forall p_1 \forall p_2 \ldots \forall p_n A$ $(n \geq 0)$. Let $\Gamma = A_1, \ldots, A_n (n \geq 0)$, then $\bigwedge \Gamma = A_1 \wedge \ldots \wedge A_n$. In particular $\bigwedge A = A$ and $\bigwedge \emptyset = \top$. $\bigvee \Gamma = \neg(\neg A_1 \wedge \ldots \wedge \neg A_n)$. In particular $\bigvee A = \neg\neg A$ and $\bigvee \emptyset = \bot$. $\neg \Gamma = \neg A_1, \ldots, \neg A_n$.

1.4 Sequent calculus

$\mathbf{LK}_{\wedge\vee\neg\exists\forall}$ has the following initial sequents and inference rules. Since $\Gamma, \Delta, \Sigma, \Pi$ are multi sets, $\mathbf{LK}_{\wedge\vee\neg\exists\forall}$ does not have the exchange rules.

$\mathbf{LJ}_{\wedge\vee\neg\exists\forall}$ is obtained from $\mathbf{LK}_{\wedge\vee\neg\exists\forall}$ by restricting right hand of sequent to at most one formula in all rules. Therefore $\mathbf{LJ}_{\wedge\vee\neg\exists\forall}$ does not have (c, r).

$\mathbf{LK}_{\wedge\neg\exists\forall}$ and $\mathbf{LJ}_{\wedge\neg\exists\forall}$ are obtained by removing $\vee$-rules from $\mathbf{LK}_{\wedge\vee\neg\exists\forall}$ and $\mathbf{LJ}_{\wedge\vee\neg\exists\forall}$ respectively.

$\mathbf{LK}_{\wedge\neg\exists}$ and $\mathbf{LJ}_{\wedge\neg\exists}$ are obtained by removing $\forall$-rules from $\mathbf{LK}_{\wedge\neg\exists\forall}$ and $\mathbf{LJ}_{\wedge\neg\exists\forall}$ respectively.

Initial sequents

$$A \Rightarrow A \qquad \Rightarrow \top \qquad \bot \Rightarrow$$

Inference rules

$$\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\,(\wedge, l)_1 \qquad \frac{A, \Gamma \Rightarrow \Delta}{B \wedge A, \Gamma \Rightarrow \Delta}\,(\wedge, l)_2 \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\,(\wedge, r)$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\,(\vee, l) \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B}\,(\vee, r)_1 \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, B \vee A}\,(\vee, r)_2$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}\ (\neg, l) \qquad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}\ (\neg, r)$$

$$\frac{A, \Gamma \Rightarrow \Delta}{\exists p A, \Gamma \Rightarrow \Delta}\ (\exists, l) \qquad \frac{\Gamma \Rightarrow \Delta, A[B/p]}{\Gamma \Rightarrow \Delta, \exists p A}\ (\exists, r) \qquad \frac{A[B/p], \Gamma \Rightarrow \Delta}{\forall p A, \Gamma \Rightarrow \Delta}\ (\forall, l) \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \forall p A}\ (\forall, r)$$

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ (w, l) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\ (w, r) \qquad \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ (c, l) \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\ (c, r)$$

$$\frac{A, \Gamma \Rightarrow \Delta}{A', \Gamma \Rightarrow \Delta}\ (name, l) \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A'}\ (name, r) \qquad \frac{\Gamma \Rightarrow \Delta, A \quad A, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (cut)$$

In $(name, l)$ and $(name, r)$, A' is obtained by replacing a bound variable p in A by other variable q.

$A[B/p]$ is the formula obtained from A by replacing all the free occurrences of p in A by the formula B, avoiding the clash of variables by applying $(name, l)$ and $(name, r)$.

In $(\exists, l)$ and $(\forall, r)$, p is not occuring as a free variable in the lower sequent.

2 Equivalency of $\mathbf{LK}_{\wedge\neg\exists}$ and $\mathbf{LJ}_{\wedge\neg\exists}$

Formulas of this section do not contain $\vee$ or $\forall$.

Definition 1 (Valuation) A *valuation* v is a mapping from the set of propositional variables to $\{\mathrm{T}, \mathrm{F}\}$. For each v, we define a mapping M_v from the set of formulas to $\{\mathrm{T}, \mathrm{F}\}$ as follows.

- $\mathrm{M}_v(p) = \mathrm{T} \iff v(p) = \mathrm{T}$
- $\mathrm{M}_v(\top) = \mathrm{T}$
- $\mathrm{M}_v(\bot) = \mathrm{F}$
- $\mathrm{M}_v(A \wedge B) = \mathrm{T} \iff \mathrm{M}_v(A) = \mathrm{T}$ and $\mathrm{M}_v(B) = \mathrm{T}$
- $\mathrm{M}_v(\neg A) = \mathrm{T} \iff \mathrm{M}_v(A) = \mathrm{F}$
- $\mathrm{M}_v(\exists p A) = \mathrm{T} \iff \mathrm{M}_v(A[\top/p]) = \mathrm{T}$ or $\mathrm{M}_v(A[\bot/p]) = \mathrm{T}$

Definition 2 M_v is extended to a mapping from the set of sequents to $\{\mathrm{T}, \mathrm{F}\}$ as
$\mathrm{M}_v(\Gamma \Rightarrow \Delta) = \mathrm{T} \iff \mathrm{M}_v(\neg(\bigwedge \Gamma) \veebar \bigvee \Delta) = \mathrm{T}$

Lemma 3 (Soundness of $\mathbf{LK}_{\wedge\neg\exists}$) If $\mathbf{LK}_{\wedge\neg\exists} \vdash\Rightarrow A$, then $\mathrm{M}_v(A) = \mathrm{T}$ for all v.

Proof. We show "If $\mathbf{LK}_{\wedge\neg\exists} \vdash \Gamma \Rightarrow \Delta$, then $\mathrm{M}_v(\Gamma \Rightarrow \Delta) = \mathrm{T}$ for all v". This is shown by induction on the height of the proof of $\Gamma \Rightarrow \Delta$.

■

Definition 4 For each $S(= \text{“}\Gamma \Rightarrow \Delta\text{”})$,

- $\mathrm{FV}(S) = \{p \mid p \text{ is occuring in } S \text{ as a free variable}\}$
- $v_i \sim_S v_j \iff v_i(p) = v_j(p)$ for all $p \in \mathrm{FV}(S)$
- $\bar{v}_i^S = \{v_j \mid v_i \sim_S v_j\}$
- $V(A\backslash S) = \{\bar{v}_i^S \mid \mathrm{M}_{v_i}(\bigwedge(\Gamma - A)) = \mathrm{T} \text{ and } \mathrm{M}_{v_i}(\bigvee \Delta) = \mathrm{F}\}$
- $V(S/A) = \{\bar{v}_i^S \mid \mathrm{M}_{v_i}(\bigwedge \Gamma) = \mathrm{T} \text{ and } \mathrm{M}_{v_i}(\bigvee(\Delta - A)) = \mathrm{F}\}$

S of $\bar{v}_i^S$ is omitted when it is obvious. $\Gamma - A$ is defined by removing one formula A from Γ. For example, if $\Gamma = \{A, A, B\}$, then $\Gamma - A = \{A, B\}$.

From now on, S_0 denotes a sequent such that $\mathrm{M}_v(S_0) = \mathrm{T}$ for all v. We will construct a tree called S_0-tree, whose nodes are associated with sequents. For each node α, the associated sequent is written as $S(\alpha)$.

Definition 5 (S_0-tree) In the begining, we make only one node α_0 which satisfies $S(\alpha_0) = S_0$. After that, we iterate applying the following table. We chose an arbitrary leaf node α. If $S(\alpha)$ matches the line i, then we add a new node α' to α and $S(\alpha')$ is defined by the line i. In the line 4, we add also α'' to α. If $S(\alpha)$ matches more than one line, we apply the line of the smallest number.

	If $S(\alpha)$ matches	$S(\alpha')$ is defined by	$S(\alpha'')$ is defined by
1	$\neg A, \Gamma \Rightarrow \Delta$	$\Gamma \Rightarrow \Delta, A$	
2	$\Gamma \Rightarrow \Delta, \neg A$	$A, \Gamma \Rightarrow \Delta$	
3	$A \wedge B, \Gamma \Rightarrow \Delta$	$A, B, \Gamma \Rightarrow \Delta$	
4	$\Gamma \Rightarrow \Delta, A \wedge B$	$\Gamma \Rightarrow \Delta, A$	$\Gamma \Rightarrow \Delta, B$
5	$\exists p A, \Gamma \Rightarrow \Delta$	$A[q_\alpha/p], \Gamma \Rightarrow \Delta$	
6	$\Gamma \Rightarrow \Delta, \exists p A$	$\Gamma \Rightarrow \Delta, A[G_\alpha/p]$	

In the line 5, q_α is a fresh variable. In the line 6, G_α is defined as follows. Let $\mathrm{FV}(S(\alpha)) = \{p_1, ..., p_n\}$. $\bar{v}$ deontes $\bar{v}^{S(\alpha)}$. We define $g_i^{\bar{v}}$ and $G_{\bar{v}}$ for each v and then we define G_α by using them.

- $g_i^{\bar{v}} = \begin{cases} p_i & (\mathrm{M}_v(p_i) = \mathrm{T}) \\ \neg p_i & (\mathrm{M}_v(p_i) = \mathrm{F}) \end{cases}$

- $G_{\bar{v}} = \begin{cases} g_1^{\bar{v}} \wedge \dots \wedge g_n^{\bar{v}} & (\mathrm{M}_v(A[\top/p]) = \mathrm{T}) \\ \perp & (\mathrm{M}_v(A[\top/p]) = \mathrm{F}) \end{cases}$
- $G_\alpha = \bigvee \{G_{\bar{v}} | \bar{v} \in V(S(\alpha)/\exists pA)\}$

The length of G_α is finite since $|V(S(\alpha)/\exists pA)| \leq 2^n$.
In this way, we add a new node until all leaves consists of only prime formulas.

Lemma 6 The construction of the S_0-tree always terminates.

Proof. This is shown by double induction on the number of $\exists$ and the number of logical symbols in the sequent.
In the line 1,2,3,4 of Definition 5, the number of $\exists$ does not change and the number of logical symbols decreases. In the line 5,6 of Definition 5, the number of $\exists$ decreases.

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Lemma 7 For all v and for all node α in the S_0-tree, $\mathrm{M}_v(S(\alpha)) = \mathrm{T}$.

Proof. This is shown by induction on definition of S_0-tree. If α is root node, then obviously $S_0(= S(\alpha))$ satisfies $\mathrm{M}_v(S_0) = \mathrm{T}$ for all v. Otherwise, we divide cases according to the lines of Definition 5.

1. ... 4. It is easy to show if $\mathrm{M}_v(S(\alpha)) = \mathrm{T}$, then $\mathrm{M}_v(S(\alpha')) = \mathrm{T}$.

5. This is the case of $S(\alpha) = \exists pA, \Gamma \Rightarrow \Delta$ and $S(\alpha') = A[q_\alpha/p], \Gamma \Rightarrow \Delta$.

   Suppose $\mathrm{M}_v(S(\alpha)) = \mathrm{T}$ for all v.

   - If $\bar{v}_i \in V(\exists pA \backslash S(\alpha))$,
     $\mathrm{M}_{v_i}(\bigwedge \Gamma) = \mathrm{T}$ and $\mathrm{M}_{v_i}(\bigvee \Delta) = \mathrm{F}$.($\because$ definition of $V(\exists pA \backslash S(\alpha))$)
     $\mathrm{M}_{v_i}(\exists pA, \Gamma \Rightarrow \Delta) = \mathrm{T}$.($\because$ i.h.)
     Therefore $\mathrm{M}_{v_i}(\exists pA) = \mathrm{F}$.
     By definition of valuation, $\mathrm{M}_{v_i}(A[\top/p]) = \mathrm{M}_{v_i}(A[\perp/p]) = \mathrm{F}$.
     This means $\mathrm{M}_{v_i}(A[q_\alpha/p]) = \mathrm{F}$ regardless of valuation of q_α. So, $\mathrm{M}_{v_i}(S(\alpha')) = \mathrm{T}$.
   - Otherwise($\bar{v}_i \notin V(\exists pA \backslash S(\alpha))$), By definition of $V(\exists pA \backslash S(\alpha))$, $\mathrm{M}_{v_i}(\bigwedge \Gamma) = \mathrm{F}$ or $\mathrm{M}_{v_i}(\bigvee \Delta) = \mathrm{T}$. Then $\mathrm{M}_{v_i}(\Gamma \Rightarrow \Delta) = \mathrm{T}$. Therefore $\mathrm{M}_{v_i}(S(\alpha')) = \mathrm{T}$

   Consequently $\mathrm{M}_v(S(\alpha')) = \mathrm{T}$ for all v.

6. This is the case of $S(\alpha) = \Gamma \Rightarrow \Delta, \exists pA$ and $S(\alpha') = \Gamma \Rightarrow \Delta, A[G_\alpha/p]$.

   If $\bar{v}_i \notin V(S(\alpha)/\exists pA)$, then $\bar{v}_i$ obviously satisfies $\mathrm{M}_{v_i}(S(\alpha')) = \mathrm{T}$. Now, we discuss about only the case of $\bar{v}_i \in V(S(\alpha)/\exists pA)$.

   First, we show:

(a) If $\mathrm{M}_{v_i}(A[\top/p]) = \mathrm{T}$, then $\mathrm{M}_{v_i}(G_\alpha) = \mathrm{T}$.

(b) If $\mathrm{M}_{v_i}(A[\top/p]) = \mathrm{F}$, then $\mathrm{M}_{v_i}(G_\alpha) = \mathrm{F}$.

- Proof of (a).

  By definition of $G_{\bar{v}}$ and each $g_k^{\bar{v}}$, $\mathrm{M}_{v_i}(G_{\bar{v}_i}) = \mathrm{M}_{v_i}(g_1^{\bar{v}_i} \wedge \ldots \wedge g_n^{\bar{v}_i}) = \mathrm{T}$. Therefore $\mathrm{M}_{v_i}(G_\alpha) = \mathrm{T}$.
- Proof of (b).

  We show $\mathrm{M}_{v_i}(G_{\bar{v}_j}) = \mathrm{F}$ for each $\bar{v}_j \in V(S(\alpha)/\exists pA)$.
  - If $\bar{v}_i = \bar{v}_j$, then $G_{\bar{v}_i} = \bot$ and $\mathrm{M}_{v_i}(G_{\bar{v}_i}) = \mathrm{F}$.
  - Otherwise($\bar{v}_i \neq \bar{v}_j$),
    * If $G_{\bar{v}_j} = \bot$, $\mathrm{M}_{v_i}(G_{\bar{v}_j}) = \mathrm{F}$
    * Otherwise($G_j = g_1^{\bar{v}_j} \wedge \ldots \wedge g_n^{\bar{v}_j}$), some $p_k \in \mathrm{FV}(S(\alpha))$ satisfies $M_{v_j}(p_k) \neq \mathrm{M}_{v_i}(p_k)(\because \bar{v}_i \neq \bar{v}_j)$. $\mathrm{M}_{v_i}(g_k^{\bar{v}_j}) = \mathrm{F}$ therefore $\mathrm{M}_{v_i}(G_{\bar{v}_j}) = \mathrm{F}$.

  Therefore $\mathrm{M}_{v_i}(G_{\bar{v}_j}) = \mathrm{F}$ for all $G_{\bar{v}_j}$. This means $\mathrm{M}_{v_i}(G_\alpha) = \mathrm{F}$.

This is a proof of (a) and (b).

On the other hand, $\mathrm{M}_{v_i}(\bigwedge \Gamma) = \mathrm{T}$ and $\mathrm{M}_{v_i}(\bigvee \Delta) = \mathrm{F}(\because$ definition of $V(S(\alpha)/\exists pA))$.
By i.h., $\mathrm{M}_{v_i}(\Gamma \Rightarrow \Delta, \exists pA) = \mathrm{T}$. Therefore $\mathrm{M}_{v_i}(\exists pA) = \mathrm{T}$. This means

($\dagger$) $\mathrm{M}_{v_i}(A[\top/p]) = \mathrm{T}$ or $\mathrm{M}_{v_i}(A[\bot/p]) = \mathrm{T}$.

Now, we show $\mathrm{M}_{v_i}(A[G_\alpha/p]) = \mathrm{T}$ for all $\bar{v}_i \in V(S(\alpha)/\exists pA)$.

- If $\mathrm{M}_{v_i}(A[\top/p]) = \mathrm{T}$, then $\mathrm{M}_{v_i}(A[G_\alpha/p]) = \mathrm{M}_{v_i}(A[\top/p]) = \mathrm{T}.(\because$ (a))
- Otherwise ($\mathrm{M}_{v_i}(A[\top/p]) = \mathrm{F}$), then $\mathrm{M}_{v_i}(A[\bot/p]) = \mathrm{T}.(\because$ ($\dagger$))

  By (b), $\mathrm{M}_{v_i}(G_\alpha) = \mathrm{F}$. Therefore $\mathrm{M}_{v_i}(A[G_\alpha/p]) = \mathrm{M}_{v_i}(A[\bot/p]) = \mathrm{T}$.

Consequently $\mathrm{M}_{v_i}(A[G_\alpha/p]) = \mathrm{T}$ for all $\bar{v}_i \in V(S(\alpha)/\exists pA)$. Then $\mathrm{M}_{v_i}(S(\alpha')) = \mathrm{T}$ for all v.

■

Lemma 8 For all $S(\alpha)$(= "$\Gamma \Rightarrow \Delta$") in the S_0-tree, $\mathbf{LJ}_{\wedge\neg\exists} \vdash S^*(\alpha)$(= "$\Gamma, \neg\Delta \Rightarrow$ ").

Proof.

This is shown by induction on maximum length from $S(\alpha)$ to the leaves in S_0-tree.

- If α is a leaf,

  $S(\alpha)$ consists of only prime formulas. By Lemma 7, $\mathrm{M}_v(S(\alpha)) = \mathrm{T}$ for all v, then at least one of the following holds.

  - $\bot \in \Gamma$.
  - $\top \in \Delta$.

– Some p satisfies $p \in \Gamma \cap \Delta$.

By applying $(w,l), (\neg,l)$ to an initial sequent, $S^*(\alpha)$ is provable in $\mathbf{LJ}_{\wedge\neg\exists}$.

- Otherwise(α is not a leaf),

  the following numbers correspond to the lines of definition 5.

  1. If $S(\alpha) = \neg A, \Gamma \Rightarrow \Delta$, then $S^*(\alpha) = S^*(\alpha')$. By i.h., we get $\mathbf{LJ}_{\wedge\neg\exists} \vdash S^*(\alpha)$.
  2. If $S(\alpha) = \Gamma \Rightarrow \Delta, \neg A$,

$$\dfrac{\dfrac{\dfrac{i.h.(S^*(\alpha'))}{A, \Gamma, \neg\Delta \Rightarrow}}{\Gamma, \neg\Delta \Rightarrow \neg A}\;(\neg, r)}{S^*(\alpha) = \neg\neg A, \Gamma, \neg\Delta \Rightarrow}\;(\neg, l)$$

  3. If $S(\alpha) = A \wedge B, \Gamma \Rightarrow \Delta$,

$$\dfrac{\dfrac{\dfrac{\dfrac{i.h.(S^*(\alpha'))}{A, B, \Gamma, \neg\Delta \Rightarrow}}{A, A \wedge B, \Gamma, \neg\Delta \Rightarrow}\;(\wedge, l)}{A \wedge B, A \wedge B, \Gamma, \neg\Delta \Rightarrow}\;(\wedge, l)}{S^*(\alpha) = A \wedge B, \Gamma, \neg\Delta \Rightarrow}\;(c, l)$$

  4. If $S(\alpha) = \Gamma \Rightarrow \Delta, A \wedge B$,

$$\dfrac{\dfrac{\dfrac{\dfrac{i.h.(S^*(\alpha'))}{\Gamma, \neg\Delta, \neg A \Rightarrow}}{\Gamma, \neg\Delta \Rightarrow \neg\neg A}\;(\neg, r) \qquad \dfrac{\dfrac{i.h.(S^*(\alpha''))}{\Gamma, \neg\Delta, \neg B \Rightarrow}}{\Gamma, \neg\Delta \Rightarrow \neg\neg B}\;(\neg, r)}{\Gamma, \neg\Delta \Rightarrow \neg\neg A \wedge \neg\neg B}\;(\wedge, r) \qquad \dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \quad B \Rightarrow B}{A, B \Rightarrow A \wedge B}\;(w,l),(\wedge,r)}{A, B, \neg(A \wedge B) \Rightarrow}\;(\neg, l)}{\neg\neg A, \neg\neg B, \neg(A \wedge B) \Rightarrow}\;(\neg,r)\times 2, (\neg,l)\times 2}{\neg\neg A \wedge \neg\neg B, \neg(A \wedge B) \Rightarrow}\;(\wedge,l)\times 2, (c,l)}{S^*(\alpha) = \Gamma, \neg\Delta, \neg(A \wedge B) \Rightarrow}\;(cut)$$

  5. If $S(\alpha) = \exists pA, \Gamma \Rightarrow \Delta$,

$$\dfrac{\dfrac{\dfrac{i.h.(S^*(\alpha'))}{A[q_\alpha/p], \Gamma, \neg\Delta \Rightarrow}}{\exists q_\alpha A[q_\alpha/p], \Gamma, \neg\Delta \Rightarrow}\;(\exists, l)}{S^*(\alpha) = \exists pA, \Gamma, \neg\Delta \Rightarrow}\;(name, l)$$

  6. If $S(\alpha) = \Gamma \Rightarrow \Delta, \exists pA$,

$$\dfrac{\dfrac{\dfrac{\dfrac{A[G_\alpha/p] \Rightarrow A[G_\alpha/p]}{A[G_\alpha/p] \Rightarrow \exists pA}\;(\exists, r)}{A[G_\alpha/p], \neg\exists pA \Rightarrow}\;(\neg, l)}{\neg\exists pA \Rightarrow \neg A[G_\alpha/p]}\;(\neg, r) \qquad \dfrac{i.h.(S^*(\alpha'))}{\Gamma, \neg\Delta, \neg A[G_\alpha/p] \Rightarrow}}{S^*(\alpha) = \Gamma, \neg\Delta, \neg\exists pA \Rightarrow}\;(cut)$$

■

Theorem 9 (Equivalency of $\mathbf{LK}_{\wedge\neg\exists}$ and $\mathbf{LJ}_{\wedge\neg\exists}$)

$$\mathbf{LK}_{\wedge\neg\exists} \vdash\Rightarrow A \quad\Longleftrightarrow\quad \mathbf{LJ}_{\wedge\neg\exists} \vdash\Rightarrow A$$

Proof.

($\Leftarrow$)

This is obvious.

($\Rightarrow$)

Suppose $\mathbf{LK}_{\wedge\neg\exists} \vdash\Rightarrow A$. By Lemma 3, $\mathrm{M}_v(\Rightarrow A) = \mathrm{T}$ for all v. Therefore we can construct ($\Rightarrow A$)-tree. $\mathbf{LJ}_{\wedge\neg\exists} \vdash\Rightarrow A$ is shown by induction on maximum length from ($\Rightarrow A$) to the leaves in ($\Rightarrow A$)-tree.

1. If A is prime formula, ($\Rightarrow A$) is a leaf of tree and ($\Rightarrow A$) is an initial sequent(This means $A = \top$). ($\Rightarrow \top$) is also an initial sequent of $\mathbf{LJ}_{\wedge\neg\exists}$.

2. If $A = B \wedge C$,

   ($\Rightarrow A$) is a parent of ($\Rightarrow B$) and ($\Rightarrow C$)($\because$ 4 of Definition 5). By i.h., $\mathbf{LJ}_{\wedge\neg\exists} \vdash\Rightarrow B$ and $\mathbf{LJ}_{\wedge\neg\exists} \vdash\Rightarrow C$. Applying $(\wedge, r)$ to ($\Rightarrow B$) and($\Rightarrow C$), we get $\mathbf{LJ}_{\wedge\neg\exists} \vdash\Rightarrow B \wedge C$.

3. If $A = \exists pB$,

   ($\Rightarrow A$) is a parent of ($\Rightarrow B[G_\alpha/p]$)($\because$ 6 of Definition 5). By i.h., $\mathbf{LJ}_{\wedge\neg\exists} \vdash\Rightarrow B[G_\alpha/p]$. Applying $(\exists, r)$ to ($\Rightarrow B[G_\alpha/p]$), we get $\mathbf{LJ}_{\wedge\neg\exists} \vdash\Rightarrow \exists pB$.

4. If $A = \neg B$,

   ($\Rightarrow A$) is a parent of ($B \Rightarrow$)($\because$ 2 of Definition 5). Applying Lemma 8 to ($B \Rightarrow$), we get $\mathbf{LJ}_{\wedge\neg\exists} \vdash B \Rightarrow$. By $(\neg, r)$, $\mathbf{LJ}_{\wedge\neg\exists} \vdash\Rightarrow \neg B$.

■

Corollary 10 Glivenko's theorem also holds in $\mathbf{LK}_{\wedge\neg\exists}$ and $\mathbf{LJ}_{\wedge\neg\exists}$. That is

$$\mathbf{LK}_{\wedge\neg\exists} \vdash \Gamma \Rightarrow A \quad\Longleftrightarrow\quad \mathbf{LJ}_{\wedge\neg\exists} \vdash \Gamma \Rightarrow \neg\neg A$$

Proof.

($\Leftarrow$)

This is obvious.

($\Rightarrow$)

Suppose $\mathbf{LK}_{\wedge\neg\exists} \vdash S_0(= \Gamma \Rightarrow A)$. By Lemma 3, $\mathrm{M}_v(S_0) = \mathrm{T}$ for all v. Therefore we can construct S_0-tree. By Lemma 8, $S_0^*(= \Gamma, \neg A \Rightarrow)$ is provable in $\mathbf{LJ}_{\wedge\neg\exists}$. Applying $(\neg, r)$ to S_0^*, we get $\mathbf{LJ}_{\wedge\neg\exists} \vdash \Gamma \Rightarrow \neg\neg A$.

■

Remark The following extension of theorem 9 does not hold.

$\mathbf{LK}_{\wedge\neg\exists} \vdash \Gamma \Rightarrow A \quad\Longleftrightarrow\quad \mathbf{LJ}_{\wedge\neg\exists} \vdash \Gamma \Rightarrow A$

A counterexample is $\neg\neg p \Rightarrow p$.

3 Equivalency of $\mathbf{LK}_{\wedge\neg\exists\forall}$ and $\mathbf{LJ}_{\wedge\neg\exists\forall}$

Formulas of this section do not contain $\vee$.

Definition 11 (Valuation) The following definition is added to Definition 1.

- $\mathrm{M}_v(\forall p A) = \mathrm{T} \iff \mathrm{M}_v(A[\top/p]) = \mathrm{T}$ and $\mathrm{M}_v(A[\bot/p]) = \mathrm{T}$

Lemma 12 (Soundness of $\mathbf{LK}_{\wedge\neg\exists\forall}$) If $\mathbf{LK}_{\wedge\neg\exists\forall} \vdash\Rightarrow A$, then $\mathrm{M}_v(A) = \mathrm{T}$ for all v.

Proof. Similar to Lemma 3.

■

Definition 13 A sequent $\Gamma \Rightarrow \Delta$ which satisfies the following conditions is called *semiprime sequent.*

- Γ consists of prime formulas.
- Δ consists of formulas whose form are $\forall \bar{p} Q$ where Q is a prime formula.

Example $p, q \Rightarrow p, \forall p(p), \forall p \forall q(r)$ is semiprime sequent.

Definition 14 (S_0-tree) Similarly to Definiton 5, S_0-tree is constructed according to the following table. We add a new node until all leaves become semiprime sequents.

	If $S(\alpha)$ matches	$S(\alpha')$ is defined by	$S(\alpha'')$ is defined by
1	$\neg A, \Gamma \Rightarrow \Delta$	$\Gamma \Rightarrow \Delta, A$	
2	$\Gamma \Rightarrow \Delta, \forall \bar{p} \neg A$	$A^+, \Gamma \Rightarrow \Delta$	
3	$A \wedge B, \Gamma \Rightarrow \Delta$	$A, B, \Gamma \Rightarrow \Delta$	
4	$\Gamma \Rightarrow \Delta, \forall \bar{p}(A \wedge B)$	$\Gamma \Rightarrow \Delta, \forall \bar{p} A$	$\Gamma \Rightarrow \Delta, \forall \bar{p} B$
5	$\exists p A, \Gamma \Rightarrow \Delta$	$A[q_\alpha/p], \Gamma \Rightarrow \Delta$	
6	$\Gamma \Rightarrow \Delta, \forall \bar{p} \exists p_{n+1} A$	$\Gamma \Rightarrow \Delta, \forall p(A[G'_\alpha/p_{n+1}])$	
7	$\forall p A, \Gamma \Rightarrow \Delta$	$A[H_\alpha/p], \Gamma \Rightarrow \Delta$	

In the line 5, q_α is a fresh variable. We define A^+, G'_α and H_α in the table as follows.

- $A^+ = A[q_1/p_1]...[q_n/p_n]$. ($q_1, ..., q_n$ are fresh variables)
- We define G'_α by,

$$\dfrac{\dfrac{S(\alpha) = \Gamma \Rightarrow \Delta, \forall \bar{p} \exists p_{n+1} A \qquad \dfrac{\dfrac{\exists p_{n+1}(A^+) \Rightarrow \exists p_{n+1}(A^+)}{\forall \bar{q} \exists p_{n+1}(A^+) \Rightarrow \exists p_{n+1}(A^+)}\,(\forall, l) \times n}{\forall \bar{p} \exists p_{n+1} A \Rightarrow \exists p_{n+1}(A^+)}\,(name, l)}{\dfrac{\Gamma \Rightarrow \Delta, \exists p_{n+1}(A^+)}{\dfrac{\Gamma \Rightarrow \Delta, A^+[G_\alpha/p_{n+1}]}{\Gamma \Rightarrow \Delta, \forall \bar{q}(A^+[G_\alpha/p_{n+1}])}\,(\forall, r) \times n}\,(\spadesuit)}\,(cut)}{S(\alpha') = \Gamma \Rightarrow \Delta, \forall \bar{p}(A[G'_\alpha/p_{n+1}])}\,(name, r)$$

(♠) is an application of the line 6 of Definition 5. We get G'_α by replacing q_i by p_i of G_α in the last $(name, r)$ rule.

- For each $\bar{v} \in V(\forall pA \backslash S(\alpha))$, we define following formula. Let $\mathrm{FV}(S(\alpha)) = \{p_1, ..., p_n\}$.

$$H_{\bar{v}} = \begin{cases} \bot & (\mathrm{M}_v(A[\top/p]) = \mathrm{T}) \\ h_1^{\bar{v}} \wedge ... \wedge h_n^{\bar{v}} & (\mathrm{M}_v(A[\top/p]) = \mathrm{F}) \end{cases}$$

These $h_i^{\bar{v}}$ are defined by

$$h_i^{\bar{v}} = \begin{cases} p_i & (\mathrm{M}_v(p_i) = \mathrm{T}) \\ \neg p_i & (\mathrm{M}_v(p_i) = \mathrm{F}) \end{cases}$$

Then H_α is defined by

$$H_\alpha = \bigvee \{H_{\bar{v}} | \bar{v} \in V(\forall pA \backslash S(\alpha))\}$$

Lemma 15 The construction of the S_0-tree always terminates.

Proof. This is shown by double induction on the number of quantifiers and the number of logical symbols in the sequents.

In the line 1,2,3,4 of Definition 14, the number of quantifiers does not increase and the number of logical symbols decreases. In the line 5,6,7 of Definition 14, the number of quantifiers decreases.

■

Lemma 16 For all node α in the S_0-tree and for all v, $\mathrm{M}_v(S(\alpha)) = \mathrm{T}$.

Proof. This is shown by induction on definition of tree.

1. If $S(\alpha) = \neg A, \Gamma \Rightarrow \Delta$, it is similar to Lemma 7.

2. If $S(\alpha) = \Gamma \Rightarrow \Delta, \forall\bar{p}\neg A$, it is shown by the following partial proof and soundness of inference rules of $\mathbf{LK}_{\wedge\neg\exists\forall}$.

$$\dfrac{\dfrac{\dfrac{i.h.(S(\alpha))}{\Gamma \Rightarrow \Delta, \forall\bar{p}\neg A} \quad \dfrac{\dfrac{\neg A^+ \Rightarrow \neg A^+}{\forall\bar{q}\neg A^+ \Rightarrow \neg A^+}(\forall, l) \times n}{\forall\bar{p}\neg A \Rightarrow \neg A^+}(name, l) \times n}{\Gamma \Rightarrow \Delta, \neg A^+}(cut) \quad \dfrac{A^+ \Rightarrow A^+}{A^+, \neg A^+ \Rightarrow}(\neg, l)}{S(\alpha') = A^+, \Gamma \Rightarrow \Delta}(cut)$$

3. If $S(\alpha) = A \wedge B, \Gamma \Rightarrow \Delta$, it is similar to Lemma 7.

4. If $S(\alpha) = \Gamma \Rightarrow \Delta, \forall\bar{p}(A \wedge B)$, it is shown by the following partial proof and soundness of inference rules of $\mathbf{LK}_{\wedge\neg\exists\forall}$.

$$\dfrac{\dfrac{i.h.(S(\alpha))}{\Gamma \Rightarrow \Delta, \forall\bar{p}(A \wedge B)} \quad \dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \wedge B \Rightarrow A}(\wedge, l)}{\forall\bar{p}(A \wedge B) \Rightarrow A}(\forall, l) \times n}{\forall\bar{p}(A \wedge B) \Rightarrow \forall\bar{p}A}(\forall, r) \times n}{S(\alpha') \; = \; \Gamma \Rightarrow \Delta, \forall\bar{p}A}(cut)$$

The case of $S(\alpha'')$ is similar.

5. If $S(\alpha) = \exists pA, \Gamma \Rightarrow \Delta$, it is similar to Lemma 7.

6. If $S(\alpha) = \Gamma \Rightarrow \Delta, \exists pA$, it is trivial because of definition of G'_α.

7. Similar to Lemma 7.

■

Lemma 17 For all $S(\alpha)$(= "$\Gamma \Rightarrow \Delta$") in the S_0-tree, $\mathbf{LJ}_{\wedge\neg\exists\forall} \vdash S^*(\alpha)$(= "$\Gamma, \neg\Delta \Rightarrow$ ").

Proof.

This is shown by induction on maximum length from $S(\alpha)$ to the leaves in S_0-tree.

- If α is leaf, then $S(\alpha)$ is semiprime formula. Obviously

  $\mathrm{M}_v(\forall\bar{p}(q)) = \mathrm{M}_v(q)$ for all $v \iff q \notin \{p_1, ..., p_n\}$.

  Since $\mathrm{M}_v(S(\alpha)) = \mathrm{T}$ for all v, at least one of the following conditions holds.

  - $\bot \in \Gamma$
  - $\top \in \Delta$
  - Some q satisfies $q \in \Gamma$ and $\forall\bar{p}(q) \in \Delta$ and $q \notin \{p_1, ..., p_n\}$

  On the other hand, following sequents are provable in $\mathbf{LJ}_{\wedge\neg\exists\forall}$

  - $\bot \Rightarrow$
  - $\neg\top \Rightarrow$
  - $q, \neg\forall\bar{p}(q) \Rightarrow$ where $q \notin \{p_1, ..., p_n\}$

  We can show $\mathbf{LJ}_{\wedge\neg\exists\forall} \vdash S^*(\alpha)$ by appling (w, l) to above sequents.

- If α is not leaf,

  1. if $S(\alpha) = \neg A, \Gamma \Rightarrow \Delta$ and $S(\alpha') = \Gamma \Rightarrow \Delta, A$, then it is similar to Lemma 8.

2. if $S(\alpha) = \Gamma \Rightarrow \Delta, \forall\bar{p}\neg A$ and $S(\alpha') = A^{+}, \Gamma \Rightarrow \Delta$, then it is shown by

$$\dfrac{\dfrac{\dfrac{\dfrac{i.h.(S^{*}(\alpha'))}{\Gamma, A^{+}, \neg\Delta \Rightarrow}}{\Gamma, \exists\bar{q}A^{+}, \neg\Delta \Rightarrow}\,(\exists, l)\times n}{\Gamma, \neg\Delta \Rightarrow \neg\exists\bar{q}A^{+}}\,(\neg, r) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A^{+} \Rightarrow A^{+}}{A^{+} \Rightarrow \exists\bar{q}A^{+}}\,(\exists, r)\times n}{A^{+}, \neg\exists\bar{q}A^{+} \Rightarrow}\,(\neg, l)}{\neg\exists\bar{q}A^{+} \Rightarrow \neg A^{+}}\,(\neg, r)}{\neg\exists\bar{q}A^{+} \Rightarrow \forall\bar{q}\neg A^{+}}\,(\forall, r)\times n}{\neg\exists\bar{q}A^{+} \Rightarrow \forall\bar{p}\neg A}\,(name, r)\times n}{S^{*}(\alpha) = \Gamma, \neg\Delta, \neg\forall\bar{p}\neg A \Rightarrow}\,(cut)$$

3. If $S(\alpha) = A \wedge B, \Gamma \Rightarrow \Delta$ and $S(\alpha') = A, B, \Gamma \Rightarrow \Delta$, then it is similar to Lemma 8.

4. If $S(\alpha) = \Gamma \Rightarrow \Delta, \forall\bar{p}(A \wedge B)$ and $S(\alpha') = \Gamma \Rightarrow \Delta, \forall\bar{p}A$, $S(\alpha'') = \Gamma \Rightarrow \Delta, \forall\bar{p}B$, then it is shown by

$$\dfrac{\dfrac{\dfrac{\dfrac{i.h.(S^{*}(\alpha'))}{\Gamma, \neg\Delta, \neg\forall\bar{p}A \Rightarrow}}{\Gamma, \neg\Delta \Rightarrow \neg\neg\forall\bar{p}A}\,(\neg, r) \quad \dfrac{\dfrac{i.h.(S^{*}(\alpha''))}{\Gamma, \neg\Delta, \neg\forall\bar{p}B \Rightarrow}}{\Gamma, \neg\Delta \Rightarrow \neg\neg\forall\bar{p}B}\,(\neg, r)}{\Gamma, \neg\Delta, \Rightarrow \neg\neg\forall\bar{p}A \wedge \neg\neg\forall\bar{p}B}\,(\wedge, r) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \quad B \Rightarrow B}{A, B \Rightarrow A \wedge B}\,(w, l), (\wedge, r)}{\forall\bar{p}A, \forall\bar{p}B \Rightarrow A \wedge B}\,(\forall, l)\times 2n}{\forall\bar{p}A, \forall\bar{p}B \Rightarrow \forall\bar{p}(A \wedge B)}\,(\forall, r)\times n}{\forall\bar{p}A, \forall\bar{p}B, \neg\forall\bar{p}(A \wedge B) \Rightarrow}\,(\neg, l)}{\neg\neg\forall\bar{p}A, \neg\neg\forall\bar{p}B, \neg\forall\bar{p}(A \wedge B) \Rightarrow}\,(\neg, l), (\neg, r)}{\neg\neg\forall\bar{p}A \wedge \neg\neg\forall\bar{p}B, \neg\forall\bar{p}(A \wedge B) \Rightarrow}\,(\wedge, l), (c, l)}{S^{*}(\alpha) = \Gamma, \neg\Delta, \neg\forall\bar{p}(A \wedge B) \Rightarrow}\,(cut)$$

5. If $S(\alpha) = \exists pA, \Gamma \Rightarrow \Delta$ and $S(\alpha') = A[q_{\alpha}/p], \Gamma \Rightarrow \Delta$, then it is similar to Lemma 8.

6. If $S(\alpha) = \Gamma \Rightarrow \Delta, \forall\bar{p}\exists p_{n+1}A$ and $S(\alpha') = \Gamma \Rightarrow \Delta, \forall pA[G'_{\alpha}/p_{n+1}]$, then it is shown by

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A[G'_{\alpha}/p_{n+1}] \Rightarrow A[G'_{\alpha}/p_{n+1}]}{A[G'_{\alpha}/p_{n+1}] \Rightarrow \exists p_{n+1}A}\,(\exists, r)}{\forall\bar{p}(A[G'_{\alpha}/p_{n+1}]) \Rightarrow \exists p_{n+1}A}\,(\forall, l)\times n}{\forall\bar{p}(A[G'_{\alpha}/p_{n+1}]) \Rightarrow \forall\bar{p}\exists p_{n+1}A}\,(\forall, r)\times n}{\neg\forall\bar{p}\exists p_{n+1}A, \forall\bar{p}(A[G'_{\alpha}/p_{n+1}]) \Rightarrow}\,(\neg, l)}{\neg\forall\bar{p}\exists p_{n+1}A \Rightarrow \neg\forall\bar{p}(A[G'_{\alpha}/p_{n+1}])}\,(\neg, r) \qquad \dfrac{i.h.(S^{*}(\alpha'))}{\Gamma, \neg\Delta, \neg\forall\bar{p}(A[G'_{\alpha}/p_{n+1}]) \Rightarrow}}{S^{*}(\alpha) = \Gamma, \neg\Delta, \neg\forall\bar{p}\exists p_{n+1}A \Rightarrow}\,(cut)$$

7. If $S(\alpha) = \forall pA, \Gamma \Rightarrow \Delta$ and $S(\alpha') = A[H_{\alpha}/p], \Gamma \Rightarrow \Delta$, then it is shown by

$$\dfrac{\dfrac{i.h.(S^{*}(\alpha'))}{A[H_{\alpha}/p], \Gamma, \neg\Delta \Rightarrow}}{S^{*}(\alpha) = \forall pA, \Gamma, \neg\Delta \Rightarrow}\,(\forall, l)$$

■

Theorem 18 (Equivalency of $\mathbf{LK}_{\wedge\neg\exists\forall}$ and $\mathbf{LJ}_{\wedge\neg\exists\forall}$)

$$\mathbf{LK}_{\wedge\neg\exists\forall} \vdash\Rightarrow A \quad \Longleftrightarrow \quad \mathbf{LJ}_{\wedge\neg\exists\forall} \vdash\Rightarrow A$$

Proof.
($\Leftarrow$)
This is obvious.
($\Rightarrow$)
this is shown in a way similar to Theorem 9 as follows.

1. If $A = \forall\bar{p}(\top)$,

   Applying $(\forall, r)$ to $\mathbf{LJ}_{\wedge\neg\exists\forall} \vdash \Rightarrow \top$, we get $\mathbf{LJ}_{\wedge\neg\exists\forall} \vdash \Rightarrow A$.

2. If $A = \forall\bar{p}(B \wedge C)$,

   $(\Rightarrow A)$ is a parent of $(\Rightarrow \forall\bar{p}B)$ and $(\Rightarrow \forall\bar{p}C)$($\because$ the line 5 of Definition 14).

   Since i.h., $\mathbf{LJ}_{\wedge\neg\exists\forall} \vdash \Rightarrow \forall\bar{p}B$ and $\mathbf{LJ}_{\wedge\neg\exists\forall} \vdash \Rightarrow \forall\bar{p}C$.

   Applying $(\wedge, r)$ to these, we get $\mathbf{LJ}_{\wedge\neg\exists\forall} \vdash \Rightarrow \forall\bar{p}B \wedge \forall\bar{p}C$.

   On the other hand, $\mathbf{LJ}_{\wedge\neg\exists\forall} \vdash \forall\bar{p}B \wedge \forall\bar{p}C \Rightarrow \forall\bar{p}(B \wedge C)$

   Applyng (cut) to these, we get $\mathbf{LJ}_{\wedge\neg\exists\forall} \vdash \Rightarrow A$.

3. If $A = \forall\bar{p}\exists p_{n+1}B$,

   $(\Rightarrow A)$ is a parent of $(\Rightarrow \forall\bar{p}(B[G'_\alpha/p_{n+1}]))$($\because$ the line 7 of Definition 14).

   Since i.h., $\mathbf{LJ}_{\wedge\neg\exists} \vdash \Rightarrow \forall\bar{p}(B[G'_\alpha/p_{n+1}])$.

   On the other hand, $\mathbf{LJ}_{\wedge\neg\exists\forall} \vdash \forall\bar{p}(B[G'_\alpha/p_{n+1}]) \Rightarrow \forall\bar{p}\exists p_{n+1}B$.

   Applying (cut) to these, we get $\mathbf{LJ}_{\wedge\neg\exists\forall} \vdash \Rightarrow A$.

4. If $A = \forall\bar{p}\neg B$,

   $(\Rightarrow A)$ is a parent of $(B^+ \Rightarrow)$($\because$ the line 3 of Definition 14).

   Applying Lemma 8 to $(B^+ \Rightarrow)$, we get $\mathbf{LJ}_{\wedge\neg\exists\forall} \vdash B^+ \Rightarrow$.

   By $(\neg, r), (\forall, r), (name, r)$, $\mathbf{LJ}_{\wedge\neg\exists\forall} \vdash \Rightarrow \forall\bar{p}\neg B$.

■

Corollary 19 (Glivenko's Theorem) Glivenko's theorem also holds in $\mathbf{LK}_{\wedge\neg\exists\forall}$ and $\mathbf{LJ}_{\wedge\neg\exists\forall}$. That is

$$\mathbf{LK}_{\wedge\neg\exists\forall} \vdash \Gamma \Rightarrow A \quad \Longleftrightarrow \quad \mathbf{LJ}_{\wedge\neg\exists\forall} \vdash \Gamma \Rightarrow \neg\neg A$$

Proof. Similar to Corollary 10.

■

Remark In the first order predicate logic, theorem 18 does not hold. A counter example is $\neg(\forall x\neg\neg P(x) \wedge \neg\forall x P(x))$. This is provable in **LK**, but not in **LJ**.

4 Partial Equivalency of $\mathbf{LK}_{\wedge\vee\neg\exists\forall}$ and $\mathbf{LJ}_{\wedge\vee\neg\exists\forall}$

Definition 20 (Valuation) The following definition is added to Definition 11.

- $\mathrm{M}_v(A \vee B) = \mathrm{T} \Longleftrightarrow \mathrm{M}_v(A) = \mathrm{T}$ or $\mathrm{M}_v(B) = \mathrm{T}$

Lemma 21 (Soundness of $\mathbf{LK}_{\wedge\vee\neg\exists\forall}$) If $\mathbf{LK}_{\wedge\vee\neg\exists\forall} \vdash \Rightarrow A$, then $\mathrm{M}_v(A) = \mathrm{T}$ for all v.

Proof. Similar to Lemma 3.

■

Definition 22 (Weak formula) *Weak formulas* are defined by

- $p, q, r, ..., \bot$ and $\top$ are weak formulas.
- $\neg A$ is a weak formula.
- $A \vee B$ is not a weak formula.
- $A \wedge B$ is a weak formula $\Longleftrightarrow$ A and B are weak formulas.
- $\exists pA$ is a weak formula $\Longleftrightarrow$ A is a weak formula.
- $\forall pA$ is a weak formula $\Longleftrightarrow$ A is a weak formula.

Example $\exists p(p \wedge \neg\neg(q \vee r))$ is a weak formula. $p \wedge \forall q(q \vee r)$ is not a weak formula.

Definition 23 For each sequent $S(=$ "$\Gamma \Rightarrow \Delta$"), we define $\mathrm{P}(S)$ and $\mathrm{N}(S)$ as the smallest set such that

- $A \in \Delta \Longrightarrow A \in \mathrm{P}(S)$
- $A \in \Gamma \Longrightarrow A \in \mathrm{N}(S)$
- $A \wedge B \in \mathrm{P}(S)$ or $A \vee B \in \mathrm{P}(S) \Longrightarrow A, B \in \mathrm{P}(S)$
- $A \wedge B \in \mathrm{N}(S)$ or $A \vee B \in \mathrm{N}(S) \Longrightarrow A, B \in \mathrm{N}(S)$
- $\exists pA \in \mathrm{P}(S)$ or $\forall pA \in \mathrm{P}(S) \Longrightarrow A \in \mathrm{P}(S)$
- $\exists pA \in \mathrm{N}(S)$ or $\forall pA \in \mathrm{N}(S) \Longrightarrow A \in \mathrm{N}(S)$
- $\neg A \in \mathrm{P}(S) \Longrightarrow A \in \mathrm{N}(S)$
- $\neg A \in \mathrm{N}(S) \Longrightarrow A \in \mathrm{P}(S)$

Example If $S = p \Rightarrow p, \neg(q \wedge r)$, then $\mathrm{P}(S) = \{p, \neg(q \wedge r)\}$ and $\mathrm{N}(S) = \{p, q, r, q \wedge r\}$.

Definition 24 (Weak sequent) A sequent S is called a *weak sequent* if S satisfies the following condition.

- $\forall pA \in \mathrm{P}(S) \Longrightarrow A$ is a weak formula.

Definition 25 (**S_0-tree**) S_0-tree of this section is obtained by adding the following lines to table of Definition 14.

	If $S(\alpha)$ matches	$S(\alpha')$ is defined by	$S(\alpha'')$ is defined by
8	$A \vee B, \Gamma \Rightarrow \Delta$	$A, \Gamma \Rightarrow \Delta$	$B, \Gamma \Rightarrow \Delta$
9	$\Gamma \Rightarrow \Delta, A \vee B$	$\Gamma \Rightarrow \Delta, A, B$	
10	$\Gamma \Rightarrow \Delta, \forall \bar{p}(A \vee B)$ $(\bar{p} \neq \emptyset)$	$\Gamma \Rightarrow \Delta, A, B$	

Lemma 26 If S_0 is a weak sequent, then all sequents of S_0-tree are also weak sequents.

proof. This is shown by induction on definition of S_0-tree.

- In the line 1,2,3,8,9,10, if $\forall pA \in \mathrm{P}(S(\alpha'))$, then $\forall pA \in \mathrm{P}(S(\alpha))$. By i.h., A is a weak formula.
- In the line 4, if $\forall pA \in \mathrm{P}(S(\alpha'))$, then $\forall pA, \forall p(A \wedge B)$ or $\forall p(B \wedge A) \in \mathrm{P}(S(\alpha))$. By i.h., $A, A \wedge B$ or $B \wedge A$ is a weak formula. Therefore A is also a weak formula.
- In the line 5, if $\forall qB \in P(S(\alpha'))$,
  - if $\forall qB$ is subformula of $C \in \Gamma \cup \Delta$, B is weak formula by i.h.
  - otherwise, $\forall qB$ is subformula of $A[q_\alpha/p]$. Since $\forall qB \in P(S(\alpha'))$, $\forall qB[p/q_\alpha] \in P(S(\alpha))$. By i.h., $B[p/q_\alpha]$ is weak formula. Then B is also weak formula.
- In the line 6, if $\forall qB \in P(S(\alpha'))$,
  - if $\forall qB$ is subformula of $C \in \Gamma \cup \Delta$, B is weak formula by i.h.
  - otherwise, $\forall qB$ is subformula of $A[H_\alpha/p]$. There is C such that $B = C[H_\alpha/p]$. Since $\forall qB \in P(S(\alpha'))$, $\forall qC \in P(S(\alpha))$. By i.h., C is weak formula. Since H_α do not contain $\vee$ or $\forall$, B is also weak formula.
- In the line 7, it is similar to the case of the line 6.

■

Lemma 27 If S_0 is a weak sequent, then the line 10 is never applied through the construction of S_0-tree.

Proof. By Lemma 26, all sequents of S_0-tree are weak sequents. But $S(\alpha)$ in the line 10 is not a weak sequent since $A \vee B$ is not weak formula and $\forall p_n(A \vee B) \in \mathrm{P}(S(\alpha))$.

■

Lemma 28 For all node α in the S_0-tree and for all v, $\mathrm{M}_v(S(\alpha)) = \mathrm{T}$.

Proof. The following cases are added to a proof of Lemma 16.

8. If $S(\alpha) = A \vee B, \Gamma \Rightarrow \Delta$, it is shown by the following partial proof and soundness of inference rules of $\mathbf{LK}_{\wedge\vee\neg\exists\forall}$.

$$\dfrac{\dfrac{A \Rightarrow A}{A \Rightarrow A \vee B}\,(\vee, r) \qquad \dfrac{i.h.(S(\alpha))}{A \vee B, \Gamma \Rightarrow \Delta}}{S(\alpha') = A, \Gamma \Rightarrow \Delta}\,(cut)$$

The case of $S(\alpha'')$ is similar.

9. If $S(\alpha) = \Gamma \Rightarrow \Delta, A \vee B$, it is shown by the following partial proof and soundness of inference rules of $\mathbf{LK}_{\wedge\vee\neg\exists\forall}$.

$$\dfrac{\dfrac{i.h.(S(\alpha))}{\Gamma \Rightarrow \Delta, A \vee B} \qquad \dfrac{A \Rightarrow A \quad B \Rightarrow B}{A \vee B \Rightarrow A, B}\,(w, r), (\vee, l)}{S(\alpha') = \Gamma \Rightarrow \Delta, A, B}\,(cut)$$

10. If $S(\alpha) = \Gamma \Rightarrow \Delta, \forall\bar{p}(A \vee B)$, it is shown by the following partial proof and soundness of inference rules of $\mathbf{LK}_{\wedge\vee\neg\exists\forall}$.

$$\dfrac{\dfrac{i.h.(S(\alpha))}{\Gamma \Rightarrow \Delta, \forall\bar{p}(A \vee B)} \qquad \dfrac{\dfrac{A \Rightarrow A \quad B \Rightarrow B}{A \vee B \Rightarrow A, B}\,(w, r), (\vee, l)}{\forall\bar{p}(A \vee B) \Rightarrow A, B}\,(\forall, l) \times n}{S(\alpha') = \Gamma \Rightarrow \Delta, A, B}\,(cut)$$

■

Lemma 29 Let S_0 a weak sequent. For all $S(\alpha)(=$ "$\Gamma \Rightarrow \Delta$") in the S_0-tree, $\mathbf{LJ}_{\wedge\vee\neg\exists\forall} \vdash S^*(\alpha)(=$ "$\Gamma, \neg\Delta \Rightarrow$ ").

Proof. The following cases are added to a proof of Lemma 17.

8. If $S(\alpha) = A \vee B, \Gamma \Rightarrow \Delta$, then it is shown by

$$\dfrac{\dfrac{i.h.(S^*(\alpha'))}{A, \Gamma, \neg\Delta \Rightarrow} \qquad \dfrac{i.h.(S^*(\alpha''))}{B, \Gamma, \neg\Delta \Rightarrow}}{S^*(\alpha) = A \vee B, \Gamma, \neg\Delta \Rightarrow}\,(\vee, l)$$

9. If $S(\alpha) = \Gamma \Rightarrow \Delta, A \vee B$, then it is shown by

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \Rightarrow A \vee B}\,(\vee, r)}{\neg(A \vee B), A \Rightarrow}\,(\neg, l)}{\neg(A \vee B) \Rightarrow \neg A}\,(\neg, r) \qquad \dfrac{\dfrac{\dfrac{B \Rightarrow B}{B \Rightarrow A \vee B}\,(\vee, r)}{\neg(A \vee B), B \Rightarrow}\,(\neg, l)}{\neg(A \vee B) \Rightarrow \neg B}\,(\neg, r)}{\neg(A \vee B) \Rightarrow \neg A \wedge \neg B}\,(\wedge, r) \qquad \dfrac{\dfrac{i.h.(S^*(\alpha'))}{\Gamma, \neg\Delta, \neg A, \neg B \Rightarrow}}{\Gamma, \neg\Delta, \neg A \wedge \neg B \Rightarrow}\,(\wedge, l) \times 2, (c, l)}{S^*(\alpha) = \Gamma, \neg\Delta, \neg(A \vee B) \Rightarrow}\,(cut)$$

10. The case that $S(\alpha) = \Gamma \Rightarrow \Delta, \forall\bar{p}(A \vee B)$ $(n \geq 1)$ is not necessary to consider since Lemma 27 holds.

■

Theorem 30 If $(\Rightarrow A)$ is weak sequent and A is a weak formula, then

$$\mathbf{LK}_{\wedge\vee\neg\exists\forall} \vdash\Rightarrow A \qquad \Longleftrightarrow \qquad \mathbf{LJ}_{\wedge\vee\neg\exists\forall} \vdash\Rightarrow A$$

Proof. Similar to Theorem 18. In the case 1,2 and 3 in the proof of Theorem 18, B and C are weak formulas and $(\Rightarrow B)$ and $(\Rightarrow C)$ are weak sequents. Since induction hypotheses are hold in all cases, this Theorem is shown similarly. The case that $A = \forall\bar{p}(B \vee C)$ is not necessary to consider since $\forall\bar{p}(B \vee C)$ is not a weak formula.

■

Corollary 31 If $(\Gamma \Rightarrow A)$ is a weak sequent, then

$$\mathbf{LK}_{\wedge\vee\neg\exists\forall} \vdash \Gamma \Rightarrow A \qquad \Longleftrightarrow \qquad \mathbf{LJ}_{\wedge\vee\neg\exists\forall} \vdash \Gamma \Rightarrow \neg\neg A$$

Proof. Similar to Corollary 19.

■

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